Limits, Derivatives, Integrals - Answer Key

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1 Limits

Evaluate the following limits. You may use l'Hopital's rule!

(a) $\lim_{x \to \infty} \frac{1}{2x+3} = \frac{1}{\infty} = 0$

(b) $\lim_{x \to 6} \frac{x-6}{|x-6|}$ DNE (calculate LHS and RHS limits)

(c) $\lim_{x \to \infty} \frac{x^4 - x^2}{2x^3 - 1} = \infty$ (l'Hopital's rule)

(d) $\lim_{x \to 0^+} \cos(x)^{\frac{1}{x}} = e^0 = 1$ (take $\ln$, use l'Hopital's rule, and exponentiate)

(e) $\lim_{x \to \infty} e^{-x} \ln(x) = 0$ ($e^{-x} = \frac{1}{e^x}$, and use l'Hopital's rule)

(f) $\lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right) = 0$ (squeeze theorem)

(g) $\lim_{x \to \infty} \sqrt{x^2 + 1} - x = 0$ (conjugate form)

(h) $\lim_{x \to 1} \frac{\sqrt{x+3} - 2}{x-1} = \frac{1}{4}$ (conjugate form, or l'Hopital's rule)

(i) $\lim_{x \to 0} \frac{e^x - 1 - x - \frac{1}{2}x^2}{x^3} = \frac{1}{6}$ (l'Hopital's rule 3 times)

(j) $\lim_{x \to -\infty} \frac{\sqrt{x^2+1}}{x} = -1$ (factor out x^2 from the square root, and $\sqrt{x^2} = |x| = -x$ since $x < 0$)

2 Derivatives

Find the derivatives of the following functions

(a) $f'(x) = e^{e^{e^x}} e^{e^x} e^x$

(b) $f'(x) = \frac{(\cos(x)+1)(\ln(x)) - (\sin(x)+x)\left(\frac{1}{x}\right)}{(\ln(x))^2}$

(c) $f'(x) = x^{\tan(x)} \left(\sec^2(x)\ln(x) + \frac{\tan(x)}{x}\right)$ (logarithmic differentiation)

(d) $f'(x) = \sec^2(\sin(\cos(2x))) \cdot \cos(-\sin(2x)) \cdot (-\sin(2x)) \cdot (2)$

(e) $y' = \frac{-(y+y^2+2xy)}{x+2xy+x^2}$

(f) $y' = \frac{7}{2}$

(g) $f'(x) = \ln(x)^{\ln(x)} \left(\frac{\ln(\ln(x))}{x} + \frac{1}{x}\right)$ (logarithmic differentiation)

(h) $f'''(x) = (x+3)e^x$

3 Integrals

Evaluate the following integrals.

(a) $\int_{-3}^{3} \sqrt{9-x^2}dx = \frac{9\pi}{2}$ (semicircle)

(b) $\int_0^2 |x-1|\, dx = 1$ (sum of areas of two triangles, draw a picture)

(c) $\int \frac{x^4+x^2}{x}dx = \frac{x^4}{4} + \frac{x^2}{2} + C$ (split up the fraction and simplify)

(d) $\int e^x\sqrt{1+e^x}dx = \frac{2}{3}\left(1+e^x\right)^{\frac{3}{2}} + C$ ($u = 1+e^x$)

(e) $\int_0^{\sqrt{\pi}} x\sin\left(x^2\right) dx = 1$ ($u = x^2$, remember to plug in the endpoints)

(f) $\int_{-\pi}^{\pi} \frac{x\cos(x)}{1+x^2}dx = 0$ (odd function!)

(g) $\frac{\int_0^{\pi} \sin(x)\cos(x)^4 dx}{\pi} = \frac{1}{5}$ ($u = \cos(x)$)

(h) $\int_e^{e^2} \frac{dx}{x\ln(x)} = \ln(2)$ ($u = \ln(x)$)

(i) $g'(x) = \ln(1+\cos(x))(-\sin(x)) - \ln(1+e^x)e^x$

(j) $\int \frac{\tan^{-1}(x)}{1+x^2}dx + \int \frac{1}{\sqrt{1-x^2}}dx = \frac{1}{2}\left(\tan^{-1}(x)\right)^2 + \sin^{-1}(x) + C$ ($u = \tan^{-1}(x)$ for the first integral)